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Polyharmonic maps into the Euclidean space

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Abstract. We study polyharmonic (k-harmonic) maps between Riemannian manifolds with finite j-energies ($j = 1, \cdots, 2k-2$). We show that if the domain is complete and the target is the Euclidean space, then such a map is harmonic.

Keywords: harmonic map, polyharmonic map, Chen's conjecture, generalized Chen's conjecture

MSC 2000 classification: primary 58E20, secondary 53C43

Introduction

This paper is an extension of our previous work ([25]) to polyharmonic maps. Harmonic maps play a central role in geometry; they are critical points of the energy functional $E(\varphi) = \frac{1}{2}\int_M |d\varphi|^2\, v_g$ for smooth maps φ of (M, g) into (N, h). The Euler-Lagrange equations are given by the vanishing of the tension filed $\tau(\varphi)$. In 1983, J. Eells and L. Lemaire [6] extended the notion of harmonic map to polyharmonic map, which are, by definition, critical points of the k-energy $(k \geq 2)$

$$E_k(\varphi) = \frac{1}{2}\int_M |(d+\delta)^k\varphi|^2\, v_g. \tag{0.1}$$

After G.Y. Jiang [15] studied the first and second variation formulas of E_2 $(k = 2)$, extensive studies in this area have been done (for instance, see [2], [4], [18], [19], [22], [26], [28], [12], [13], [14], etc.). Notice that harmonic maps are always polyharmonic by definition.

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For harmonic maps, it is well known that:

If a domain manifold (M, g) *is complete and has non-negative Ricci curvature, and the sectional curvature of a target manifold* (N, h) *is non-positive, then every energy finite harmonic map is a constant map* (cf. [29]).

In our previous paper, we showed that

Theorem 1. *[25] Let* (M, g) *be a complete Riemannian manifold, and the curvature of* (N, h) *is non-positive. Then,*

(1) *every biharmonic map* $\varphi : (M, g) \to (N, h)$ *with finite energy and finite bienergy must be harmonic.*

(2) *In the case* $\mathrm{Vol}(M, g) = \infty$*, every biharmonic map* $\varphi : (M, g) \to (N, h)$ *with finite bienergy is harmonic.*

Now, in this paper, we want to extend it to k-harmonic maps ($k \geq 2$). Indeed, we will show

Theorem 2. *Theorems 4 and 6 Let* (M, g) *be a complete Riemannian manifold, and* (N, h)*, the* n*-dimensional Euclidean space. Then,*

(1) *every* k*-harmonic map* $\varphi : (M, g) \to (N, h)$ $(k \geq 2)$ *with finite* j*-energies for all* $j = 1, 2, \cdots, 2k-2$*, must be harmonic.*

(2) *In the case of* $\mathrm{Vol}(M, g) = \infty$*, every* k*-harmonic map* $\varphi : (M, g) \to (N, h)$ *with finite* j*-energy for all* $j = 2, 4, \cdots, 2k-2$*, is harmonic.*

Theorem 2 gives an affirmative answer to the generalized B.Y. Chen's conjecture (cf. [4]) on k-harmonic maps ($k \geq 2$) under the L^2-conditions.

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1 Preliminaries and statement of main theorem

In this section, we prepare materials for the first variational formula for the biharmonic maps. Let us recall the definition of a harmonic map $\varphi : (M, g) \to (N, h)$, of a compact Riemannian manifold (M, g) into another Riemannian manifold (N, h), which is an extremal of the *energy functional* defined by

$$E(\varphi) = \int_M e(\varphi)\, v_g,$$

where $e(\varphi) := \frac{1}{2}|d\varphi|^2$ is called the energy density of φ. That is, for any variation $\{\varphi_t\}$ of φ with $\varphi_0 = \varphi$,

$$\left.\frac{d}{dt}\right|_{t=0} E(\varphi_t) = -\int_M h(\tau(\varphi), V)v_g = 0, \tag{1.1}$$

where $V \in \Gamma(\varphi^{-1}TN)$ is a variation vector field along φ which is given by $V(x) = \frac{d}{dt}|_{t=0}\varphi_t(x) \in T_{\varphi(x)}N$, $(x \in M)$, and the *tension field* is given by $\tau(\varphi) = \sum_{i=1}^m B(\varphi)(e_i, e_i) \in \Gamma(\varphi^{-1}TN)$, where $\{e_i\}_{i=1}^m$ is a locally defined frame field on (M, g), and $B(\varphi)$ is the second fundamental form of φ defined by

$$\begin{aligned} B(\varphi)(X, Y) &= (\widetilde{\nabla} d\varphi)(X, Y) \\ &= (\widetilde{\nabla}_X d\varphi)(Y) \\ &= \overline{\nabla}_X(d\varphi(Y)) - d\varphi(\nabla_X Y), \end{aligned} \tag{1.2}$$

for all vector fields $X, Y \in \mathfrak{X}(M)$. Here, ∇, and ∇^N, are the Levi-Civita connections of (M, g), (N, h), respectively, and $\overline{\nabla}$, and $\widetilde{\nabla}$ are the induced ones on $\varphi^{-1}TN$, and $T^*M \otimes \varphi^{-1}TN$, respectively. By (2), φ is harmonic if and only if $\tau(\varphi) = 0$.

The second variation formula is given as follows. Assume that φ is harmonic. Then,

$$\left.\frac{d^2}{dt^2}\right|_{t=0} E(\varphi_t) = \int_M h(J(V), V)v_g, \tag{1.3}$$

where J is an elliptic differential operator, called the *Jacobi operator* acting on $\Gamma(\varphi^{-1}TN)$ given by

$$J(V) = \overline{\Delta} V - \mathcal{R}(V), \tag{1.4}$$

where $\overline{\Delta} V = \overline{\nabla}^* \overline{\nabla} V = -\sum_{i=1}^m \{\overline{\nabla}_{e_i} \overline{\nabla}_{e_i} V - \overline{\nabla}_{\nabla_{e_i} e_i} V\}$ is the *rough Laplacian* and $\mathcal{R}$ is a linear operator on $\Gamma(\varphi^{-1}TN)$ given by $\mathcal{R}(V) = \sum_{i=1}^m R^N(V, d\varphi(e_i))d\varphi(e_i)$, and R^N is the curvature tensor of (N, h) given by $R^N(U, V) = \nabla^N{}_U \nabla^N{}_V - \nabla^N{}_V \nabla^N{}_U - \nabla^N{}_{[U,V]}$ for U, $V \in \mathfrak{X}(N)$.

J. Eells and L. Lemaire [6] proposed polyharmonic (k-harmonic) maps and Jiang [15] studied the first and second variation formulas for biharmonic maps. Let us consider the *bienergy functional* defined by

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 v_g, \tag{1.5}$$

where $|V|^2 = h(V, V)$, $V \in \Gamma(\varphi^{-1}TN)$. The first variation formula of the bienergy functional is given by

$$\left.\frac{d}{dt}\right|_{t=0} E_2(\varphi_t) = -\int_M h(\tau_2(\varphi), V)v_g. \tag{1.6}$$

Here,

$$\tau_2(\varphi) := J(\tau(\varphi)) = \overline{\Delta}(\tau(\varphi)) - \mathcal{R}(\tau(\varphi)), \tag{1.7}$$

which is called the *bitension field* of φ, and J is given in (5).

A smooth map φ of (M, g) into (N, h) is said to be *biharmonic* if $\tau_2(\varphi) = 0$.

Now let us recall the definition of the k-energy $E_k(\varphi)$ $(k \geq 2)$:

Definition 1. The *k-energy* $E_k(\varphi)$ $(k \geq 2)$ is defined formally ([7]) by

$$E_k(\varphi) := \frac{1}{2}\int_M |(d+\delta)^k \varphi|^2 \, v_g \tag{1.8}$$

for every smooth map $\varphi \in C^\infty(M, N)$. Then, it is given ([12], p. 270, Lemma 40) by the following formula:

$$E_k(\varphi) = \begin{cases} \dfrac{1}{2}\displaystyle\int_M |W_\varphi^\ell|^2 \, v_g & \text{(if k is even, say $2\,\ell$)}, \\ \dfrac{1}{2}\displaystyle\int_M |\overline{\nabla}\, W_\varphi^\ell|^2 \, v_g & \text{(if k is odd, say $2\ell+1$)}. \end{cases} \tag{1.9}$$

Here, W_φ^ℓ is given as, by definition,

$$W_\varphi^\ell := \underbrace{\overline{\Delta}\cdots\overline{\Delta}}_{\ell-1}\,\tau(\varphi) \in \Gamma(\varphi^{-1}TN). \tag{1.10}$$

For $k = 1$, that is, $\ell = 0$, we define $W_\varphi^0 = \varphi$, also.

Then, the definition and the first variation formula for the k-energy E_k are given as follows:

Definition 2. `k-harmonic map` For each $k = 2, 3, \cdots$, and a smooth map $\varphi : (M, g) \to (N, h)$, is *$k$-harmonic* if

$$\left.\frac{d}{dt}\right|_{t=0} E_k(\varphi_t) = 0 \tag{1.11}$$

for every smooth variation $\varphi_t : M \to N$ $(-\varepsilon < t < \varepsilon)$ with $\varphi_0 = \varphi$.

Then, we have ([12], p.269, Theorem 39)

Theorem 3. *`The first variation formula of the k-energy`* *Assume that $(N, h) = (\mathbb{R}^n, h_{\mathbb{R}^n})$ is the n-dimensional Euclidean space. For every $k = 2, 3, \cdots$, it holds that*

$$\left.\frac{d}{dt}\right|_{t=0} E_k(\varphi_t) = -\int_M \langle \tau_k(\varphi), V\rangle \, v_g, \tag{1.12}$$

where V is a variation vector field given by $V(x) = \frac{d}{dt}\big|_{t=0}\varphi_t(x) \in T_{\varphi(x)}N$ ($x \in M$). The k-tension field $\tau_k(\varphi)$ is given by

$$\tau_k(\varphi) = J(W_\varphi^{k-1}) = \overline{\Delta}(W_\varphi^{k-1}), \tag{1.13}$$

where $W_\varphi^{k-1} = \underbrace{\overline{\Delta}\cdots\overline{\Delta}}_{k-2}\,\tau(\varphi) \in \Gamma(\varphi^{-1}TN)$.

Thus, $\varphi : (M,g) \to (N,h)$ is k-harmonic if and only if $\overline{\Delta}^{k-1}\tau(\varphi) = 0$ which is equivalent to $W_\varphi^k = 0$.

Notice that the formula (14) of the k-tension field $\tau_k(\varphi)$ coincides with the k-tension field in Theorems 2.2 and 2.3 in [21] in the case that the target space (N,h) is the n-dimensional Euclidean space $(N,h) = (\mathbb{R}^n, h_{\mathbb{R}^n})$ because of $R^N \equiv 0$.

Here, we denote by $\overline{\nabla}W_\varphi^\ell = \overline{\nabla}\varphi = d\varphi$ for $\ell = 0$, and $k = 2\ell + 1 = 1$,

$$E_1(\varphi) = \frac{1}{2}\int_M |d\varphi|^2\, v_g.$$

Then, we can state our main theorem.

Theorem 4. `Main theorem` *Assume that the domain manifold (M,g) is a complete Riemannian manifold, and the target space (N,h) is the n-dimensional Euclidean space. Let $\varphi : (M,g) \to (N,h)$ be a k-harmonic map ($k \geq 2$). Assume that*

(1) *$E_j(\varphi) < \infty$ for all $j = 2, 4, \cdots, 2k-2$, and*

(2) *either*

$$E_j(\varphi) < \infty \text{ for all } j = 1, 3, \cdots, 2k-3, \text{ or}$$
$$\mathrm{Vol}(M,g) = \infty.$$

Then, $\varphi : (M,g) \to (N,h)$ is harmonic.

In the case of the n-dimensional Euclidean space $(N,h) = (\mathbb{R}^n, h_{\mathbb{R}^n})$, Theorem 4 and the following Theorem 5 are natural extensions of our previous theorem in [25] which is:

Theorem 5. *Assume that (M,g) is complete and the sectional curvature of (N,h) is non-positive.*

(1) *Every biharmonic map $\varphi : (M,g) \to (N,h)$ with finite energy $E(\varphi) < \infty$ and finite bienergy $E_2(\varphi) < \infty$, is harmonic.*

(2) *In the case $\mathrm{Vol}(M,g) = \infty$, every biharmonic map $\varphi : (M,g) \to (N,h)$ with finite bienergy $E_2(\varphi) < \infty$, is harmonic.*

2 The iteration proposition.

By virtue of (10), we have to notice the the energy conditions in (1) and (2) of Theorem 4:

Indeed, the condition which $E_j(\varphi) < \infty$ for all $j = 2, 4, \cdots, 2k-2$ in (1) of Theorem 4 is equivalent to that

$$\int_M |W_\varphi^j|^2 \, v_g < \infty \qquad (j = 1, 2, \cdots, k-1), \tag{2.1}$$

and the condition which $E_j(\varphi) < \infty$ for all $j = 1, 3, \cdots, 2k-3$ in (2) of Theorem 4 is equivalent to that

$$\int_M |\overline{\nabla} W_\varphi^j|^2 \, v_g < \infty \qquad (j = 0, 1, \cdots, k-2). \tag{2.2}$$

Therefore, to show Theorem 4, we only have to prove the following theorem:

Theorem 6. *Assume that the domain manifold (M, g) is a complete Riemannian manifold, and the target space (N, h) is the n-dimensional Euclidean space. Let $\varphi : (M, g) \to (N, h)$ be a k-harmonic map.*

Assume that

(1) $\displaystyle\int_M |W_\varphi^j|^2 \, v_g < \infty$ *for all* $j = 1, 2, \cdots, k-1$, *and*

(2) *either*

$$\int_M |\overline{\nabla}\, W_\varphi^j|^2 \, v_g < \infty \text{ for all } j = 0, 1, \cdots, k-2, \text{ or}$$

$$\mathrm{Vol}(M, g) = \infty.$$

Then, $\varphi : (M, g) \to (N, h)$ is harmonic.

To prove Theorem 6 whose proof will be given in the next section, we need the following iteration proposition:

Proposition 1. `the iteration method` *Let (M, g) be a complete Riemannian manifold, and (N, h), an arbitrary Riemannian manifold. Let $\varphi : (M, g) \to (N, h)$ be an arbitrary C^∞ map satisfying that for some $j \geq 2$,*

$$W_\varphi^j = 0. \tag{2.3}$$

If we assume the following two conditions:

$$\begin{cases} (1) \quad \displaystyle\int_M |W_\varphi^{j-1}|^2 \, v_g < \infty, \text{ and} \\ (2) \quad \text{either } \int_M |\overline{\nabla}\, W_\varphi^{j-2}|^2 \, v_g < \infty \text{ or } \mathrm{Vol}(M, g) = \infty, \end{cases} \tag{2.4}$$

then, we have

$$W_\varphi^{j-1} = 0. \tag{2.5}$$

Remark 1. Under the assumptions (16), if we have $W_\varphi^k = 0$ for some $k \geq 2$, then we have automatically, $W_\varphi^1 = \tau(\varphi) = 0$, i.e., φ is harmonic.

In this section, we give a proof of Proposition 1 which consists of four steps.

(*The first step*) For a fixed point $x_0 \in M$, and for every $0 < r < \infty$, we first take a cut-off C^∞ function η on M (for instance, see [16]) satisfying that

$$\begin{cases} 0 \leq \eta(x) \leq 1 & (x \in M), \\ \eta(x) = 1 & (x \in B_r(x_0)), \\ \eta(x) = 0 & (x \notin B_{2r}(x_0)), \\ |\nabla \eta| \leq \dfrac{2}{r} & (x \in M). \end{cases} \tag{2.6}$$

(*The second step*) Notice that (17) is equivalent to that

$$\overline{\Delta}\, W_\varphi^{j-1} = 0 \tag{2.7}$$

because of $W_\varphi^j = \overline{\Delta}\, W_\varphi^{j-1}$.

Then, we have

$$\begin{aligned} 0 &= \int_M \langle \eta^2\, W_\varphi^{j-1}, \overline{\Delta}\, W_\varphi^{j-1} \rangle\, v_g \\ &= \int_M \sum_{i=1}^m \langle \overline{\nabla}_{e_i}(\eta^2\, W_\varphi^{j-1}), \overline{\nabla}_{e_i} W_\varphi^{j-1} \rangle\, v_g \\ &= \int_M \eta^2 \sum_{i=1}^m |\overline{\nabla}_{e_i} W_\varphi^{j-1}|^2 v_g + 2 \int_M \sum_{i=1}^m \eta\, e_i(\eta) \langle W_\varphi^{j-1}, \overline{\nabla}_{e_i} W_\varphi^{j-1} \rangle\, v_g. \end{aligned} \tag{2.8}$$

By moving the second term in the last equality of (22) to the left hand side, we have

$$\begin{aligned} \int_M \eta^2 \sum_{i=1}^m |\overline{\nabla}_{e_i} W_\varphi^{j-1}|^2 &= -2 \int_M \sum_{i=1}^m \langle \eta\, \overline{\nabla}_{e_i} W_\varphi^{j-1}, e_i(\eta)\, W_\varphi^{j-1} \rangle\, v_g \\ &= -2 \int_M \sum_{i=1}^m \langle S_i, T_i \rangle\, v_g, \end{aligned} \tag{2.9}$$

where we put $S_i := \eta\, \overline{\nabla}_{e_i} W_\varphi^{j-1}$, and $T_i := e_i(\eta)\, W_\varphi^{j-1}$ $(i = 1 \cdots, m)$.

Now let recall the following inequality:

$$\pm 2\,\langle S_i, T_i\rangle \leq \varepsilon |S_i|^2 + \frac{1}{\varepsilon}|T_i|^2 \tag{2.10}$$

for all positive $\varepsilon > 0$ because of the inequality $0 \leq |\sqrt{\varepsilon}\,S_i \pm \frac{1}{\sqrt{\varepsilon}}\,T_i|^2$. Therefore, for (24), we obtain

$$-2\int_M \sum_{i=1}^m \langle S_i, T_i\rangle\, v_g \leq \varepsilon \int_M \sum_{i=1}^m |S_i|^2\, v_g + \frac{1}{\varepsilon}\int_M \sum_{i=1}^m |T_i|^2\, v_g. \tag{2.11}$$

If we put $\varepsilon = \frac{1}{2}$, we obtain, by (23) and (25),

$$\begin{aligned}\int_M \eta^2 \sum_{i=1}^m |\overline{\nabla}_{e_i} W_\varphi^{j-1}|^2\, v_g &\leq \frac{1}{2}\int_M \sum_{i=1}^m \eta^2\, |\overline{\nabla}_{e_i} W_\varphi^{j-1}|^2\, v_g \\ &\quad + 2\int_M \sum_{i=1}^m e_i(\eta)^2\, |W_\varphi^{j-1}|^2\, v_g. \end{aligned}\tag{2.12}$$

Thus, by (26) and (20), we obtain

$$\begin{aligned}\int_M \eta^2 \sum_{i=1}^m |\overline{\nabla}_{e_i} W_\varphi^{j-1}|^2\, v_g &\leq 4\int_M |\nabla\eta|^2\, |W_\varphi^{j-1}|^2\, v_g \\ &\leq \frac{16}{r^2}\int_M |W_\varphi^{j-1}|^2\, v_g. \end{aligned}\tag{2.13}$$

(*The third step*) By definition of η in the first step, (27) turns out that

$$\int_{B_r(x_0)} |\overline{\nabla}\, W_\varphi^{j-1}|^2\, v_g \leq \frac{16}{r^2}\int_M |W_\varphi^{j-1}|^2\, v_g. \tag{2.14}$$

Here, recall our assumption that (M, g) is complete and non-compact, and (1) $\int_M |W_\varphi^{j-1}|^2\, v_g < \infty$. When we tend $r \to \infty$, the right hand side of (26) goes to zero, and the left hand side of (26) goes to $\int_M |\overline{\nabla} W_\varphi^{j-1}|^2\, v_g$. Thus, we obtain

$$0 \leq \int_M |\overline{\nabla} W_\varphi^{j-1}|^2\, v_g \leq 0,$$

which implies that

$$\overline{\nabla}\, W_\varphi^{j-1} = 0 \tag{2.15}$$

everywhere on M.

(*The fourth step*) (a) In the case that $\int_M |\overline{\nabla}\, W_\varphi^{j-2}|^2\, v_g < \infty$, let us define a smooth 1-form α on M by

$$\alpha(X) := \langle W_\varphi^{j-1}, \overline{\nabla}_X W_\varphi^{j-2}\rangle \qquad (X \in \mathfrak{X}(M)). \tag{2.16}$$

Then, we have:

$$\mathrm{div}(\alpha) = -|W_\varphi^{j-1}|^2. \tag{2.17}$$

Because we have

$$\begin{aligned}
\mathrm{div}(\alpha) &= \sum_{i=1}^m (\nabla_{e_i}\alpha)(e_i) \\
&= \sum_{i=1}^m \{e_i(\alpha(e_i)) - \alpha(\nabla_{e_i} e_i)\} \\
&= \sum_{i=1}^m \left\{ e_i\left(\langle\, W_\varphi^{j-1}, \overline{\nabla}_{e_i} W_\varphi^{j-2}\rangle\right) - \langle W_\varphi^{j-1}, \overline{\nabla}_{\nabla_{e_i} e_i} W_\varphi^{j-2}\rangle \right\} \\
&= \sum_{i=1}^m \left\{ \langle \overline{\nabla}_{e_i} W_\varphi^{j-1}, \overline{\nabla}_{e_i} W_\varphi^{j-2}\rangle + \langle W_\varphi^{j-1}, \overline{\nabla}_{e_i}\overline{\nabla}_{e_i} W_\varphi^{j-2}\rangle \right. \\
&\qquad\qquad \left. - \langle W_\varphi^{j-1}, \overline{\nabla}_{\nabla_{e_i} e_i} W_\varphi^{j-2} \right\} \\
&= \langle W_\varphi^{j-1}, -\overline{\Delta} W_\varphi^{j-2}\rangle \qquad \text{(because of (29) and definition of } \overline{\Delta}) \\
&= -|W_\varphi^{j-1}|^2,
\end{aligned} \tag{2.18}$$

which is (31).

Furthermore, we have

$$\int_M |\alpha|\, v_g < \infty. \tag{2.19}$$

Because we have, by definition of α in (30),

$$\begin{aligned}
\int_M |\alpha|\, v_g &= \int_M |\langle W_\varphi^{j-1}, \overline{\nabla} W_\varphi^{j-2}\rangle|\, v_g \\
&\leq \left(\int_M |W_\varphi^{j-1}|^2\, v_g\right)^{\frac{1}{2}} \left(\int_M |\overline{\nabla} W_\varphi^{j-2}|^2\, v_g\right)^{\frac{1}{2}} \\
&< \infty
\end{aligned} \tag{2.20}$$

because of our assumptions $\int_M |W_\varphi^{j-1}|^2\, v_g < \infty$ and $\int_M |\overline{\nabla} W_\varphi^{j-2}|^2\, v_g < \infty$. Thus, we can apply Gaffney's theorem to this α (cf. [10], and Theorem 4.1 in

Appendix in [25]). We obtain

$$0 = \int_M \mathrm{div}(\alpha)\, v_g = -\int_M |W_\varphi^{j-1}|^2\, v_g, \tag{2.21}$$

which implies that $W_\varphi^{j-1} = 0$.

(b) In the case that $\mathrm{Vol}(M,g) = \infty$, we first notice that $|W_\varphi^{j-1}|^2$ is constant on M, say C_0. Because for every $X \in \mathfrak{X}(M)$, we have

$$X\, |W_\varphi^{j-1}|^2 = 2\, \langle \overline{\nabla}_X W_\varphi^{j-1}, W_\varphi^{j-1} \rangle = 0 \tag{2.22}$$

due to (29). Then, due to the assumption (1) of Proposition 1, and the above, we obtain

$$\infty > \int_M |W_\varphi^{j-1}|^2\, v_g = C_0 \int_M v_g = C_0\, \mathrm{Vol}(M,g). \tag{2.23}$$

By our assumption that $\mathrm{Vol}(M,g) = \infty$, (37) implies that $C_0 = 0$. We obtain $W_\varphi^{j-1} \equiv 0$. We obtain Proposition 1. $\boxed{QED}$

Proof of Theorem 6. We apply Proposition 1 to our map $\varphi : (M,g) \to (N,h)$, then the iteration procedure works well since φ is k-harmonic, i.e., $W_\varphi^k = 0$. Then, we have $W_\varphi^{k-1} = 0$, and then we have $W_\varphi^{k-2} = 0$, etc. Finally, we obtain $\tau(\varphi) = W_\varphi^1 = 0$. Thus, $\varphi : (M,g) \to (N,h)$ is harmonic. We obtain Theorem 6. $\boxed{QED}$

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